

## Definitions and Operations - ACTIVITIES

### 5. Activities

#### ACTIVITY ONE

##### Learning Objectives

**LO1: Students learn how to complete matrix multiplication**

**LO2: Students learn how to use Excel to complete matrix multiplication**

**LO3: Students learn how to use the matrix operators**

#### Task One

$$\text{If: } A = \begin{pmatrix} 3 & 6 \\ 1 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}$$

Calculate (a) AB and (b) BA.

#### Task Two

An excellent way of reinforcing students understanding of matrix multiplication is to use Excel. The appropriate function is “MMULT”. There are two key differences when using this function compared to the majority of Excel functions:

Rather than pressing “Enter” or clicking “OK” if using the function wizard you must press “CTRL”+”Shift”+”Enter” together to tell Excel to undertake the calculations.

Since the outcome of this function is usually going to be something other than a scalar you must inform Excel where to display the output. Thus if the outcome is a (2 x 1) matrix you must first of all highlight 2 rows and 1 column before starting to use the function.

Students might wish to see a video showing these operations being undertaken using Excel

[www.lawseconomics.co.uk/metal/mmult.zip](http://www.lawseconomics.co.uk/metal/mmult.zip)

#### Task Three

$$\text{If: } A = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 2 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

Calculate BA.

## Definitions and Operations - ANSWERS

### ACTIVITY ONE

#### Task One

Remembering what we have learned previously and following the helpful hint:

$$(2 \times 2) \times (2 \times 2)$$

$$(2 \times 2)$$

Therefore whichever way we perform the multiplication the result will be a  $2 \times 2$  matrix.

$$AB = \begin{pmatrix} 3 & 6 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 3 \times 3 + 6 \times 2 & 3 \times 2 + 6 \times 4 \\ 1 \times 3 + 4 \times 2 & 1 \times 2 + 4 \times 4 \end{pmatrix} = \begin{pmatrix} 21 & 30 \\ 11 & 18 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 3 \times 3 + 2 \times 1 & 3 \times 6 + 2 \times 4 \\ 2 \times 3 + 4 \times 1 & 2 \times 6 + 4 \times 4 \end{pmatrix} = \begin{pmatrix} 11 & 26 \\ 10 & 28 \end{pmatrix}$$

#### Task Two

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#### Task Three

$$BA = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 & 4 \\ 3 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 \times 2 + 3 \times 3 & 2 \times 1 + 3 \times 2 & 2 \times 4 + 3 \times 0 \\ 1 \times 2 + 4 \times 3 & 1 \times 1 + 4 \times 2 & 1 \times 4 + 4 \times 0 \end{pmatrix} = \begin{pmatrix} 13 & 8 & 8 \\ 14 & 9 & 4 \end{pmatrix}$$

## Transposing and Inverting a Matrix and Matrix Determinants - ACTIVITES

### ACTIVITY ONE

#### Learning Objectives

**LO1: Students have opportunity to refresh calculation of averages**

**LO2: Students to learn how to calculate averages using matrices**

**LO3: Students learn how matrices can be used in applied economics**

Consider the data below which are the weekly stock price returns of three UK gambling stocks:

	<b>SPORTINGBET</b>	<b>888 HOLDINGS</b>	<b>PARTYGAMING</b>
17/07/2006	-18.79	-6.65	-2.83
24/07/2006	-29.43	-18.13	1.46
31/07/2006	23.49	-1.90	4.78
07/08/2006	17.60	7.42	9.13
14/08/2006	-19.12	-9.31	-11.92
21/08/2006	4.81	-2.65	8.79
28/08/2006	-3.78	2.72	-0.87
04/09/2006	7.53	4.14	5.07
11/09/2006	-43.20	-9.70	-10.48
18/09/2006	19.10	4.93	0.00
25/09/2006	2.62	-6.71	-5.85
02/10/2006	-62.50	-22.12	-55.22

The average weekly return is:

Sporting Bet = -8.47%

888 Holdings = -4.83%

Partygaming = - 4.83%

**Task One**

Using matrices how would I find the average return of Sporting Bet?

**Task Two**

Confirm that the average of both 888 Holdings and Partygaming is -4.83%

**ACTIVITY TWO**

**Learning Objectives**

**LO1: Students learn the meaning and significance of covariance**

**LO2: Students learn how to calculate variance using matrix algebra**

**LO3: Students learn how to apply matrix algebra to solve an applied economics problem**

**LO4: Students learn how to make an economic inference using covariance information**

***Background information***

The variance of a portfolio can be calculated as:

$$\sigma_p^2 = x' \Sigma x$$

where :

$x$  is a vector of portfolio weights (wealth invested),

$\Sigma$  is a matrix of covariances between assets

For a four asset case this would look like the following when expanded out:

$$\sigma_p^2 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

where :

$x_i$  is the weight invested in asset  $i$ ,

$\sigma_{ij}$  is the covariance between asset  $i$  and  $j$ ,

Note here we first of all multiply a (1x4) row vector by a (4x4) matrix to obtain a (1x4) row vector which is multiplied by a (4x1) column vector to give a (1x1) answer.

**Task One**

Let us assume the following covariances between four UK stocks:

Covariance	MMO <sup>2</sup>	Marks and Spencer	Barclays	Scottish Power
<b>MMO<sup>2</sup></b>	27.30	2.10	3.60	-0.08
<b>Marks and Spencer</b>	2.10	15.92	3.54	1.62
<b>Barclays</b>	3.60	3.54	13.33	1.88
<b>Scottish Power</b>	-0.08	1.62	1.88	7.02

Calculate the variance of an equally weighted ( $x_i=0.25$ ) portfolio using matrix algebra

**Task Two**

What can you infer about risk?

**ACTIVITY THREE**

**Learning Objectives**

**LO1: Students learn how to find the inverse of a matrix**

**LO2: Students learn how to calculate the determinant**

**Task One**

Find the inverse of the following matrix

$$A = \begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}$$

**Task Two**

Find the determinant of the following 3 x 3 matrix:

$$|A| = \begin{vmatrix} 5 & 4 & 9 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{vmatrix}$$

**Task Three**

Find the inverse of the following matrix:

$$A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$$

**ACTIVITY FOUR****Learning Objectives**

**LO1: Students learn how to independently calculate determinants**

**LO2: Students learn how to make inferences about whether a matrix is singular or not**

**LO3:**

**Task One**

Find the determinant of the following 3x3 matrix:

$$|A| = \begin{vmatrix} 20 & 5 & 5 \\ 15 & 10 & 5 \\ 5 & 0 & 1 \end{vmatrix}$$

**Task Two**

Find the determinant of the following 4x4 matrix:

$$|A| = \begin{vmatrix} 2 & 2 & 4 & 9 \\ 4 & 1 & 0 & 2 \\ 4 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 \end{vmatrix}$$

## Transposing and Inverting a Matrix and Matrix Determinants -

### ANSWERS

#### ACTIVITY ONE

##### Task One

An average is found by adding up a series of numbers and dividing by the number that they are. So for the series above there are twelve numbers so we add them together and divide by 12. To replicate the figure of -8.47% above using matrices we would multiply:

$$\left(\frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{12}\right) \begin{pmatrix} -18.79 \\ -29.43 \\ 23.49 \\ 17.60 \\ -19.12 \\ 4.81 \\ -3.78 \\ 7.53 \\ -43.20 \\ 19.10 \\ 2.62 \\ -62.50 \end{pmatrix} = -8.47$$

##### Task Two

Students should confirm through matrix algebra that the assertion is indeed correct.

#### ACTIVITY TWO

##### Task One

$$\sigma_p^2 = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix} \begin{pmatrix} 27.30 & 2.10 & 3.60 & -0.08 \\ 2.10 & 15.92 & 3.54 & 1.62 \\ 3.60 & 3.54 & 13.33 & 1.88 \\ -0.08 & 1.62 & 1.88 & 7.02 \end{pmatrix} \begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{pmatrix}$$

Note that the diagonal elements here are the individual variances of the four stocks. We can see that MMO<sup>2</sup> is the most volatile stock. Interestingly, we also see that MMO<sup>2</sup> and Scottish Power have negative covariance.

Using the “MMULT” command in Excel we can find the answer very easily:

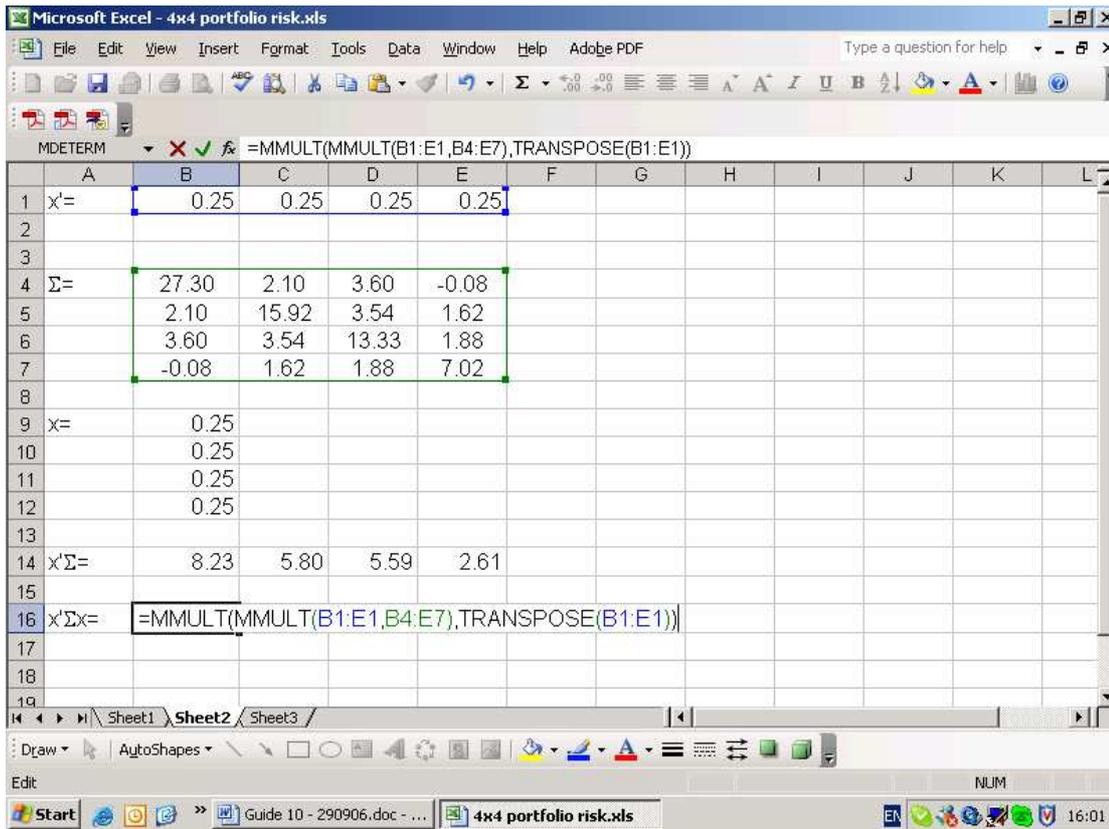
$x' =$	0.25	0.25	0.25	0.25
$\sigma =$	27.30	2.10	3.60	-0.08
	2.10	15.92	3.54	1.62
	3.60	3.54	13.33	1.88
	-0.08	1.62	1.88	7.02
$x =$	0.25			
	0.25			
	0.25			
	0.25			
$x' \sigma =$	8.23	5.80	5.59	2.61
$x' \sigma x =$	5.56			

**Task Two**

Notice that the answer of 5.56 is less than any of the individual variances and hence we can reduce risk by putting 25% of our wealth in each asset rather than “putting all our eggs in one basket”.

This spreadsheet is available at <http://www.lawseconomics.co.uk/metal/covariance.xls>

Note when carrying out these calculations using Excel it is possible to “nest” functions as follows:



**Note:** Rather than referring to an array of cells as say B4:E4 (as in the screen shot above) it is possible to name the range of cells (Using “Insert”, “Name”, “Define”). You could therefore replace B4:E4 with “sigma” in the example above.

**ACTIVITY THREE**

**Task One**

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} 3 & -5 \\ -2 & 3 \end{pmatrix} = \frac{1}{3 \times 3 - 5 \times 2} \begin{pmatrix} 3 & -5 \\ -2 & 3 \end{pmatrix} = -1 \times \begin{pmatrix} 3 & -5 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 5 \\ 2 & -3 \end{pmatrix}$$

**Task Two**

Expanding by the 3<sup>rd</sup> row (due to the presence of a zero):

$$|A| = -3 \begin{vmatrix} 5 & 9 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} = -3(5 \times 1 - 9 \times 1) + (5 \times 2 - 4 \times 1) = 12 + 6 = 18$$

**Task Three**

First of all finding the determinant by expanding by column 1:

$$\det(A) = 1 \times (16-9) - 1 \times (12 - 9) + 1 \times (9-12) = 7 - 3 - 3 = 1$$

Then finding the nine cofactors:

$$A_{11} = +(16 - 9) = 7$$

$$A_{21} = -(12 - 9) = -3$$

$$A_{31} = +(9 - 12) = -3$$

$$A_{12} = -(4 - 3) = -1$$

$$A_{22} = +(4 - 3) = 1$$

$$A_{32} = -(3 - 3) = 0$$

$$A_{13} = +(3 - 4) = -1$$

$$A_{23} = -(3 - 3) = 0$$

$$A_{33} = +(4 - 3) = 1$$

Putting these in-order forms the adjugate matrix:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

Then taking the transpose of the adjugate matrix to form the adjoint matrix:

$$\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

This was a very straight forward task due to the symmetry of the adjugate matrix.

The inverse is therefore:

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \frac{1}{|A|} \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Why have we done all this? Hopefully, students will be questioning why there has been so much emphasis on finding the determinants and inverses of matrices. In order to reinforce that it is time well spent we should use a number of examples.

### ACTIVITY FOUR

#### Task One

Expanding by the 3<sup>rd</sup> row (due to the presence of a zero):

$$|A| = +5 \begin{vmatrix} 5 & 5 \\ 10 & 5 \end{vmatrix} + 1 \begin{vmatrix} 20 & 5 \\ 15 & 10 \end{vmatrix} = +5(5 \times 5 - 5 \times 10) + (20 \times 10 - 5 \times 15) = -125 + 125 = 0$$

The value of the determinant is zero. This implies that the corresponding matrix is singular and hence has no inverse.

#### Task Two

The first thing to ask is the pattern of “+’s” and “-’s”. This is just an extension of the pattern introduced earlier:

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

The next thing to consider is what row or column is it optimal to “expand” by. If we look closely we will see that row 3 and column 3 both have two zero’s which will reduce the number of calculation. Therefore we should choose either of those.

Expanding by column 3:

$$|A| = 4 \times \begin{vmatrix} 4 & 1 & 2 \\ 4 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 2 & 9 \\ 4 & 1 & 2 \\ 4 & 1 & 0 \end{vmatrix}$$

We now have to find the determinants of the 3x3 cofactor matrices.

Consider the term before the minus sign and expanding by row 2:

$$4 \times \left\{ -4 \times \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + 1 \times \begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} \right\}$$

$$= 4 \times \{ -4 \times (1 \times 1 - 2 \times 2) + 1(4 \times 1 - 2 \times 3) \}$$

$$= 4 \times \{ -4 \times -3 + 1 \times (-2) \} = 4 \times (12 - 2) = +40$$

Now considering the term after the minus sign and expanding by row 3:

$$-1 \times \left\{ +4 \begin{vmatrix} 2 & 9 \\ 1 & 2 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 9 \\ 4 & 2 \end{vmatrix} \right\}$$

$$= -1 \times \{ +4(2 \times 2 - 9 \times 1) - 1 \times (2 \times 2 - 9 \times 4) \}$$

$$= -1 \times \{ +4(-5) - 1 \times (-32) \}$$

$$= -1 \times \{ -20 + 32 \} = -12$$

The determinant of this 4 x 4 matrix is therefore 40-12 = 28.

## Cramer's Rule - ACTIVITIES

### Learning Objectives

**LO1: Students learn the practical application of Cramer's rule to national income determination**

**LO2: Students apply their knowledge to a simple problem of evaluating national income**

### Task One

Given the following typical National Income model found in most introductory macroeconomics text books:

$$Y = C + I + G$$

$$C = bY_d$$

$$T = tY$$

Where  $Y$  = Income,  $Y_d$  = disposable income,  $C$  = Consumption,  $T$  = Taxation,  $b$  = marginal propensity to consume,  $t$  = rate of taxation,  $I$  = autonomous Investment and  $G$  = autonomous Government spending.

Hence:

Re-writing the three equations in terms of  $Y$ ,  $C$  and  $T$  we obtain:

$$Y - C = I + G$$

$$C = b(Y - T) = bY - bT \text{ and so } -bY + C + bT = 0$$

$$-tY + T = 0$$

These three equations can be represented in matrix form as:

$$\begin{pmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{pmatrix} \begin{pmatrix} Y \\ C \\ T \end{pmatrix} = \begin{pmatrix} I + G \\ 0 \\ 0 \end{pmatrix}$$

Using Cramm's rule we can solve for the equilibrium levels of  $Y$  and/or  $C$  and/or  $T$  as follows:

$$Y = \frac{\begin{vmatrix} I+G & -1 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix}} = \frac{(I+G)\begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 0 & b \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -b & b \\ -t & 1 \end{vmatrix} + 0\begin{vmatrix} -b & 1 \\ -t & 0 \end{vmatrix}} = \frac{I+G}{1+bt-b}$$

$$C = \frac{\begin{vmatrix} 1 & I+G & 0 \\ -b & 0 & b \\ -t & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix}} = \frac{1\begin{vmatrix} 0 & b \\ 0 & 0 \end{vmatrix} - (I+G)\begin{vmatrix} -b & b \\ -t & 1 \end{vmatrix} + 0\begin{vmatrix} -b & 0 \\ -t & 0 \end{vmatrix}}{\begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -b & b \\ -t & 1 \end{vmatrix} + 0\begin{vmatrix} -b & 1 \\ -t & 0 \end{vmatrix}} = \frac{(I+G)b(1-t)}{1+bt-b}$$

$$T = \frac{\begin{vmatrix} 1 & -1 & I+G \\ -b & 1 & 0 \\ -t & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix}} = \frac{1\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - (-1)\begin{vmatrix} -b & 0 \\ -t & 0 \end{vmatrix} + (I+G)\begin{vmatrix} -b & 1 \\ -t & 0 \end{vmatrix}}{\begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -b & b \\ -t & 1 \end{vmatrix} + 0\begin{vmatrix} -b & 1 \\ -t & 0 \end{vmatrix}} = \frac{(I+G)t}{1+bt-b}$$

We therefore have algebraic solutions that allow us to consider what would happen to the equilibrium levels of Y, C and T if, for example, the Government changed G or t.

**Task One**

(a) Review the above exposition of Cramer's rule.

(b) Assume that

I=300, G=200, b = 0.7 and t = 0.2.

What is the value of Y?

## Cramer's Rule - ANSWER

### Task One

(b)

$$Y = \frac{I + G}{1 + bt - b} = \frac{300 + 200}{1 + 0.7 \times 0.2 - 0.7} = \frac{500}{1 + 0.14 - 0.7} = \frac{500}{0.44} = 1136.36$$

$$C = \frac{(I + G)b(1 - t)}{1 + bt - b} = \frac{(200 + 300) \times 0.7 \times (0.8)}{0.44} = \frac{500 \times 0.56}{0.44} = 636.36$$

$$T = \frac{(I + G)t}{1 + bt - b} = \frac{(200 + 300) \times 0.2}{0.44} = \frac{100}{0.44} = 227.27$$

$$Y = C + I + G = 636.36 + 300 + 200 = 1136.36$$

## Input-Output Analysis - ACTIVITES

### Task One

#### Learning Objectives

**LO1: Students learn how to apply their knowledge of matrices to input-output analysis**

**LO2: Students learn how to draw inferences from answers derived using matrix algebra**

### Task One

Ask students to review the worked example above (See 3.). Working in pairs, students should create a Powerpoint presentation which summarises:

- what the worked example was trying to solve
- the **key** steps to solving the input-output problem; and
- any problems and difficulties they encountered when working through the worked example.

### Task Two

Working in pairs. create your own values of A and Y and assume  $n=3$ .