

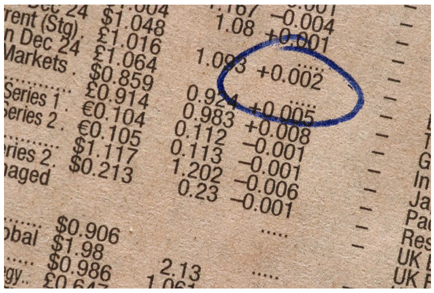

Teaching and Learning

Guide 9:

Integration

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Section 1: Introduction to the guide

Many students studying an undergraduate course in economics or business will be unfamiliar with the mathematical concept on integration. Those who have a basic grounding in integration – for example, integrating simple quadratic functions – are unlikely to have a grasp of the practical applications of integration.

The challenge then for economics lecturers then is to address these three central issues, namely:

- To help students to learn what integration means as a mathematical process;
- To create opportunities for students to practise using integration as a problem-solving technique; and
- To ensure students have an awareness of how integration can be used to analyse and solve economic problems and issues.

These challenges are formidable for many reasons. First, a proportion of students will come to their undergraduate course with limited knowledge, understanding or competency in mathematics beyond a basic level such as GCSE. Second, some students will tend to doubt their ability to learn or apply new mathematical techniques, particularly ones which they perceive as difficult. Sometimes this response is due to low levels of mathematical confidence or self-esteem. For others, it is a practical response to avoid their fear of failing. In all cases though, this offers a practical challenge to the lecturer or seminar leader.

Finally, some students can simply fail to learn mathematical techniques because they do not understand the importance or context and this is an important consideration: as education professionals we all know that if mathematics is delivered in a lofty, abstract or opaque way so it is more likely we lose students who struggle to see the practical or vocational perspective of mathematics.

This guide does not intend to provide learning materials per se. Rather, it intends to help colleagues secure very good student learning by offering a resource which helps to deliver high quality teaching. Implicitly, the guide attempts to ‘bring alive’ the topic of integration and to engender in students an appreciation of what integration is, how it is applied and how it can be used.

One strong ambition underpinning this guide is to help dispel the 'myth of calculus' and to improve the transparency and understanding of this vitally important mathematical technique. Central to this are the five keys to deliver any high quality seminar[1]:

- The capacity to take into account different abilities;
- To take into account different learning styles;
- To create opportunities to develop and embed transferable skills;
- To ensure that teaching and learning is active and student centred; and
- Scope exists for the seminar or lecture to incorporate evaluation and reflection.

Above all, if students can understand why they are learning about integration, how the technique is used and its practical applications to the modern world they are much more likely to grasp the essentials of how they might independently use it.

Section 2: The Concept of Integration

1. The concept of integration

Some students will have heard of 'calculus' and a proportion will recognise the term, 'differentiation'. Students who have not followed A-level Mathematics – or equivalent – will not have encountered integration as a topic at all and of those who have very few will have had much opportunity to gain any insight into how integration is used in any practical sense.

2. Presenting the concept of integration

It is advisable to deliver the topic of integration **after** students have fully grasped differentiation. In this way, integration can be initially introduced as 'reverse differentiation' with a simple and clear definition: A simple scenario could help to 'break the ice'. For example, an economics researcher picks up the research of an absent colleague and discovers a sheaf of working out of differentiation. She wants to know what the original functions were – that is, before they were differentiated – and integration offers her a way to do this.

[1] See "Seminars" in 'The Handbook for Economics Lecturers', Dr Rebecca Taylor

3. Delivering the concept of integration to small or larger groups

Both small and larger groups would benefit from an early contextualisation of integration i.e. to confirm to students that it has a practical and applied benefit to economists. For example, in the table below:

Integration can be used to analyse this economic issue:	By:	Students can research further by:
Calculating the volumes of products which can be manufactured in a factory.	Integrating a mathematical production function to calculate 3D solids or "volumes of revolution"	www.sussex.ac.uk/Units/economics/micro1/lectures/prodfns.doc
Estimate the quantities of land, labour and capital required to produce a product.	Integrating a production function can help to determine quantities of inputs or factors of production required.	www.sussex.ac.uk/Units/economics/micro1/lectures/prodfns.doc
Income distribution and how it might have changed over time.	Calculating the Gini coefficient: the area between the line of 'perfect income equality' and the line of 'actual income distribution'	http://en.wikipedia.org/wiki/Gini_coefficient
Calculating consumer surplus and informing analyses regarding consumer welfare and competition policy.	Calculating the area below the demand curve but above the equilibrium price level.	https://www.tutor2u.net/economics/content/topics/marketsinaction/consumer_surplus.htm
Working out the present value of a future stream of income or revenue e.g. an annuity	Calculating the value of an investment over a given time period.	http://www.investopedia.com/terms/n/npv.asp

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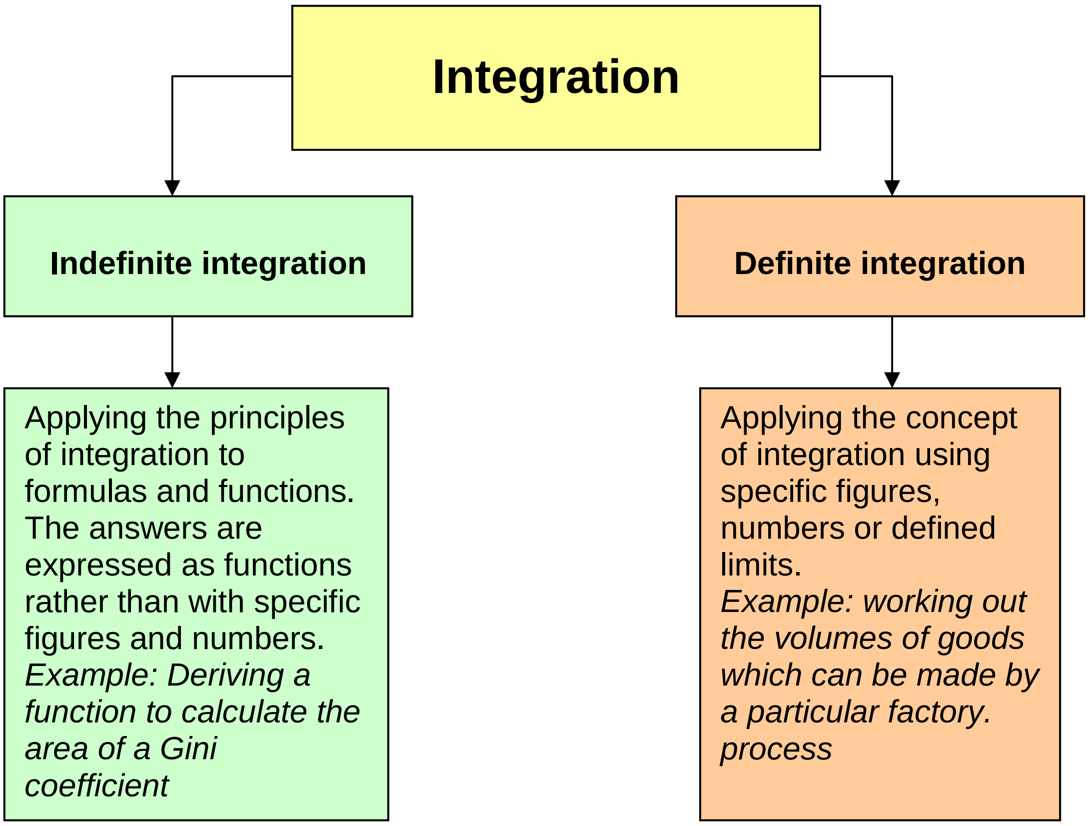
4. Discussion questions

Students could be asked to share their research findings on uses of integration (See above) short presentations. This could prompt some discussion n the usefulness and practical applications of integration. Students should be encouraged to refer to other personal experiences or understanding of integration e.g. a student following a natural science might have used integration to calculate displacement or particle acceleration.

Section 3: Indefinite Integrals

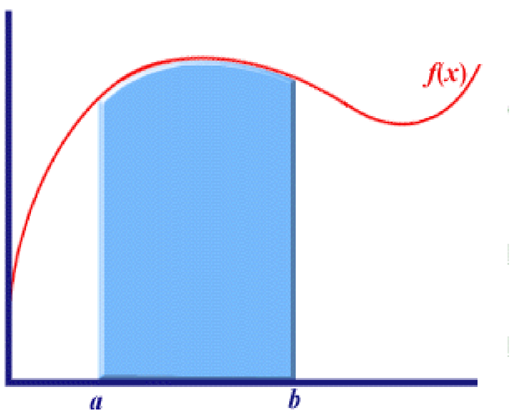
1. The concept of indefinite integrals

Perhaps one of the most effective ways to start this material is to explain the difference between ‘indefinite’ and ‘definite’ integrals. The language here is a little off-putting and students are likely to benefit from a simple and concise distinction such as:



2. Presenting the concept of indefinite integrals

An obvious and direct way to show indefinite integrals is to use the example of calculating the area under a graph e.g.



Students might need to be reminded of the meaning of the term *f*(x)

Students could be encouraged to think of functions and curves that they have encountered in their economics course and to think of any examples where it might be useful to calculate the area underneath the curve. Some students could extend this to include any areas between curves e.g. consumer surplus. Some examples are given below which might be useful as 'prompts'.

Economic Function or Variable	**Describe the area**	**Reason why this area might be useful to an economist**
The Demand Curve	Underneath the demand curve but above the price line	This measures consumer welfare, a measure of consumer welfare.
The Supply Curve	The area below the supply curve and below the equilibrium price.	This area measures the transfer earnings: the amount that a factor of production must earn to prevent it moving or transferring to an alternative use.
The Demand for a Good X, the domestic supply curve for Good X and the world supply curve for Good X.	The (horizontal) distance between the domestic demand and supply curves and the (vertical) distance between the world and domestic price levels for the Good X.	The area calculates the value of Good X which the domestic economy will import.
Fixed capital formation or Investment	Using the accelerator theory of investment, we can work out the area or stock of capital needed.	This could be useful for firms planning future investment decisions. This principle could be used for large scale public sector investment e.g. NHS or capital spending on road building.

Understanding and Using Common Notation

Students will need to understand the key terms used when using integration. Students might benefit from creating their own “Integration Glossary” which they can complete either from lecture notes or from other independent research. An example is set out below with suggested student descriptions of key notation in italics:

Expression or Symbol	Write in words what this means
$\int$	*This symbol represents integration and looks like an elongated S (which stands for "sum").*
$y = f(x)$	*This means that the variable y depends upon – or is a function of – another variable x.*
$\int f(x)\,dx$	*This is an indefinite integral. You can tell because there are no limits on the top or bottom of the ∫.*
$\int f(x)\,dx + c$	*The ‘c’ is the constant of integration. This is the figure which disappeared when the function was differentiated. We cannot know for sure what the original value for ’c’ was but we need to take account of it in our integrated function.*

3. Delivering the concept of indefinite integrals to small or larger groups

Students will need opportunities to practise integrating simple functions.

Students will need to be clear of the general form:

If :

$$\frac{dy}{dx} = \alpha x^{\beta}$$

$$\text{then} \quad : y = \left(\frac{\alpha x^{\beta+1}}{\beta + 1} \right) + c \quad \text{as long as } \beta \neq -1$$

$$\Rightarrow \int \alpha x^{\beta} \, dx = \left(\frac{\alpha x^{\beta+1}}{\beta + 1} \right) + c \quad \text{as long as } \beta \neq -1$$

An effective and straightforward way for students in small groups to practice integration is to give them opportunities to take turns constructing equations and then for another group to discuss and agree the integral (See Task One below). Students could be encouraged to work in pairs on creating a spreadsheet which would calculate simple in integrals (again see below).

Larger groups might benefit from a summary Powerpoint presentation which shows in a simple and animated way exactly how a function can be integrated. Students could be asked to create their own Powerpoint which they could use to test their own understanding and also to serve as a revision tool or aide memoire.

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4. Discussion questions

Students could discuss ways in which they would teach the concept of integration if they were the lecturer. This could raise issues concerning how they feel they learn and could help the lecturer design the next series of seminars.

5. Activities

Task One

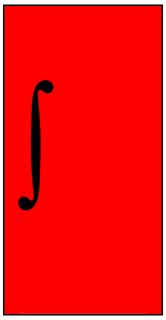
Learning Objectives

LO1: To understand the link between a differentiated function and an integrated function

LO2: To be able to confidently and independently integrate simple functions

Students are divided into an even number of groupings typically of 3 or 4 students in each group. Each group has a number of A4 laminated cards comprising of the following:

y	dx	c	x	f(x)
+	-	x	÷	=
0	1	2	3	4
5	6	7	8	9
0	1	2	3	4
5	6	7	8	9

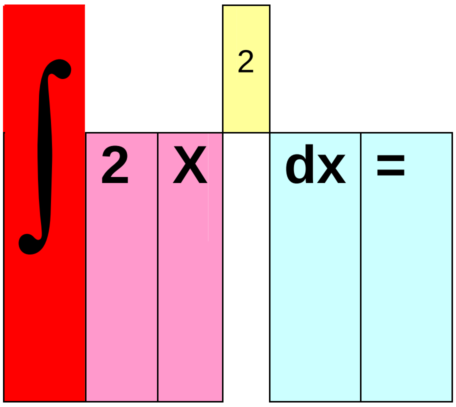


The light yellow cards are superscripted numbers and allow powers to be used. A separate and thinner or narrower card is used for the ∫ sign. This allows the figures and variables to be placed next to the integration sign.

One group – the "question setters" - then agrees a function and sets this out using the laminated cards. The other group - the "responders" then has 60 seconds to discuss and agree what the integral of this function is and to then set this out using the laminated cards.

For example:

Group 1: Question Setters



Group 2: Responders

The group would then use their set of cards to, hopefully, create the correct answer i.e. $\frac{2}{3}x^3 + c$

This activity offers numerous benefits. It helps students to understand the structure of the integral – that is, the integral sign, then the function then the "dx" element – and this is reinforced by the use of coloured cards. The activity also offers an active learning approach to integration and this could be extended by asking students to move between different groups and creating more kinaesthetic learning. And, of course, there are also an infinite number of questions and answers which students can create themselves.

Task Two

LO1: To be able to confidently and independently integrate simple functions

LO2: To be able to explain why a particular answer is correct.

Part A

Students are given a prepared sheet of indefinite integration problems and asked to map questions with answers. This activity can be particularly useful for students who may not be the most confident at calculus: in effect, they are given the mapping of answers and they are therefore implicitly guided or supported to find the correct solution.

A typical set of questions and answers is set out below together with the answers:

Integration problem	Answer
$\int 4x^5 dx$	$\frac{x^3}{3}+c$
$\int x^2 dx$	$\frac{-3}{x^2}+c$
$\int \frac{6}{x^3} dx$	$\frac{2x^6}{3}+c$
$\int 12x \ \ dx$	$\frac{x^3}{3}+\frac{x^2}{2}+c$
$\int x(x+1)\,dx$	$\frac{x^3}{3}+\frac{1}{x}+c$
$\int \left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)dx$	$6x^2+c$

Part B

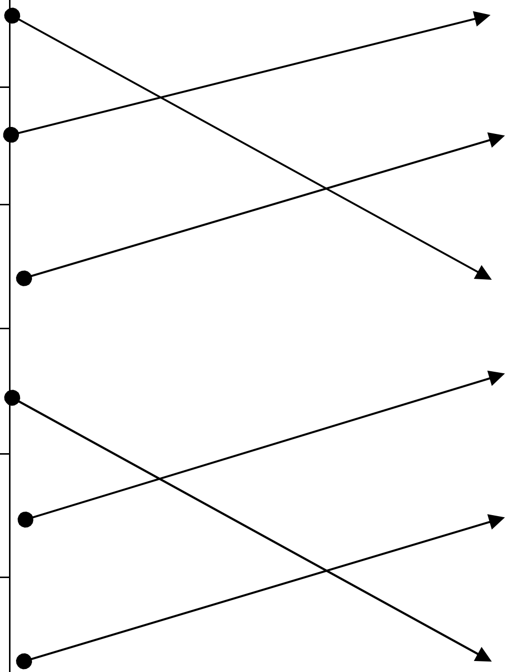
The Part B tasks are the more 'traditional' indefinite integration problems. Higher ability students could think of ways this could be calculated using an Excel spreadsheet and the website http://www.webmath.com/integrate.html might help students to think about how they could produce something similar.

Find:
$\int x^2 dx$
$\int x^5 dx$
$\int \left(\frac{1}{x^2}\right) dx$
$\int a^{-7} \, da$
$\int 7 \quad \mathrm{dx}$
$\int \left(\frac{5x^2}{\sqrt{x}}\right) dx$
$\int 2x^2(3-4x)\, dx$

ANSWERS

Part A

Integration problem
$\int 4x^5 dx$
$\int x^2 dx$
$\int \frac{6}{x^3} dx$
$\int 12x \quad dx$
$\int x(x+1)\,\mathrm{dx}$
$\int \left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)dx$



Answer
$\frac{x^3}{3}+c$
$\frac{-3}{x^2}+c$
$\frac{2x^6}{3}+c$
$\frac{x^3}{3}+\frac{x^2}{2}+c$
$\frac{x^3}{3}+\frac{1}{x}+c$
$6x^2+c$

Part B

Find:	ANSWER
$\int x^2 \quad dx$	$\frac{1}{3}x^3+c$
$\int x^5 dx$	$\frac{1}{6}x^6+c$
$\int \left(\frac{1}{x^2}\right)dx$	$-\frac{1}{x}$
$\int a^{-7} da$	$-\frac{a^{-6}}{6}$ **or** $\frac{-1}{6a^6}$
$\int 7 \quad \mathrm{dx}$	$7x+c$

Find:	ANSWER
$\int \left(\frac{5x^2}{\sqrt{x}} \right) dx$	$2x^{\frac{5}{2}} + c$
$\int 2x^2(3-4x)\,dx$	$2x^3 - 2x^4 + c$

6. Top Tips

It is easy with a topic such as integration to simply teach students the technique and then expect them to fire off answers to a set of standard and uninspiring problem sets. This is particularly true for something like indefinite integrals which can seem rather detached or abstract for many students.

There are opportunities for students to work together in contexts such as ‘questioning and responding’ (see above) and also in discussing how integration can be used in modern economics.

The learning of higher ability groups also need to be catered for. These students could be encouraged to undertake further research on advanced integration topics e.g. logarithmic or using trigonometry. We need to be mindful that some students will have completed Advanced Mathematics courses already.

7. Conclusion

In some ways the work students cover on indefinite integrals sets the tone and their outlook on the material which follows: definite integrals and integration using exponentials. It is therefore vital that at this initial stage students have chances to practice these mathematical techniques, to solve problems in pairs and independently and to understand the practical relevance of in integration

Section 4: Definite Integrals

1. The concept of definite integrals

Students will probably find it easier to move on to the concept of definite integrals once they have fully grasped the idea of indefinite integrals and had opportunities to practice solving them.

In some ways, indefinite integrals can be seen as a way to introduce the 'theory' with definite integrals offering more scope for lecturers and students to apply this knowledge to real world problems. Colleagues will want to explain to students the reason for structuring their learning in this way.

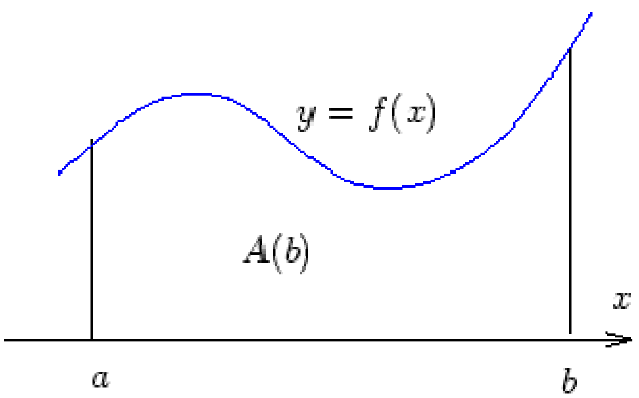
2. Presenting the concept of definite integrals

There are a range of strategies to effectively introduce the concept of definite integrals.

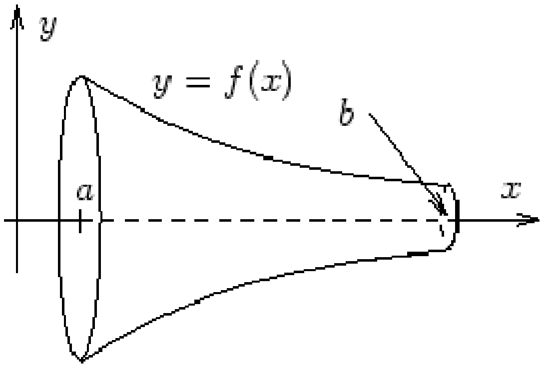
Strategy 1: Graphical illustration

Two suggestions are shared here. The first would involve the lecturer projecting the graph of a simple mathematical function. A quadratic would probably be the easiest and most straightforward to use e.g. $y=3x^2-7x+4$. The lecturer can then explain that integration can be used to calculate the area under the graph between two points.

For example,



The second strategy develops this point and is likely to be more suited to higher ability students. Students could learn about definite integrals by seeing how this approach can generate volumes of revolution. For example, the diagram below could be used to show that the 2D area under the function y=f(x) can be used to generate the 3D 'chimney' or 'funnel'



This approach would be particularly useful for the higher ability students.

Strategy 2: Formal solution

This second general teaching and learning strategy would probably be best used after Strategy 1. This simply involves taking students through the formal process of solving a definite integral using the general expression:

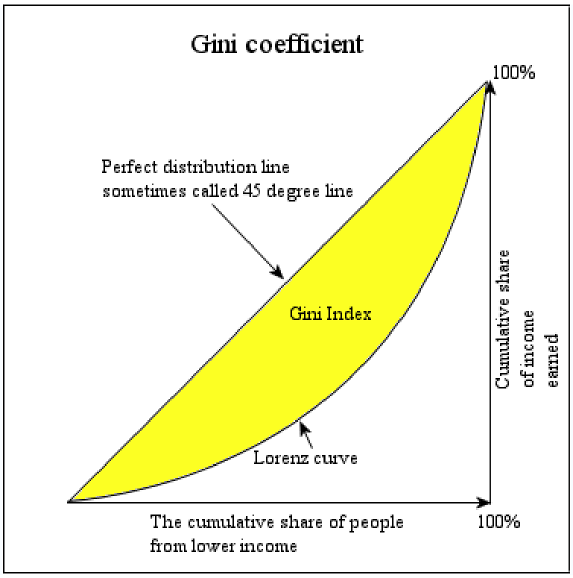
$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

Where: a is the lower limit

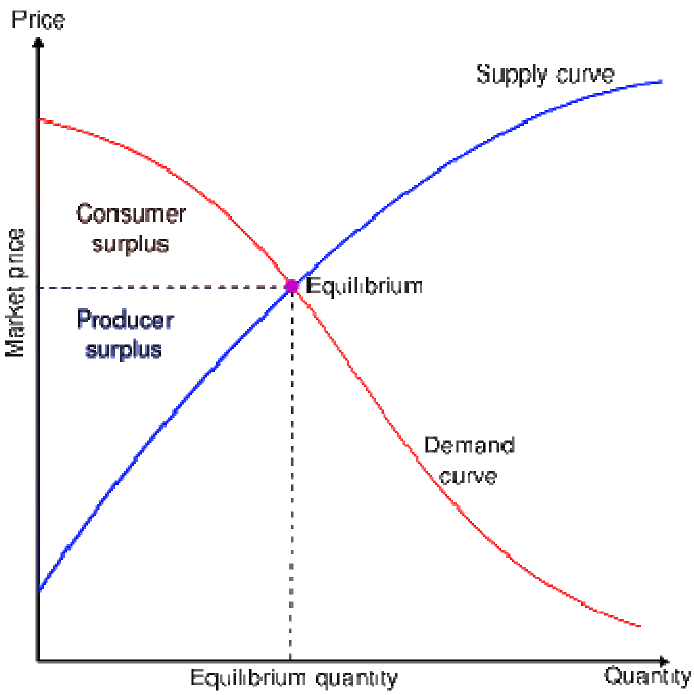
b is the upper limit

Strategy 3: Direct Application to an Economics Problem

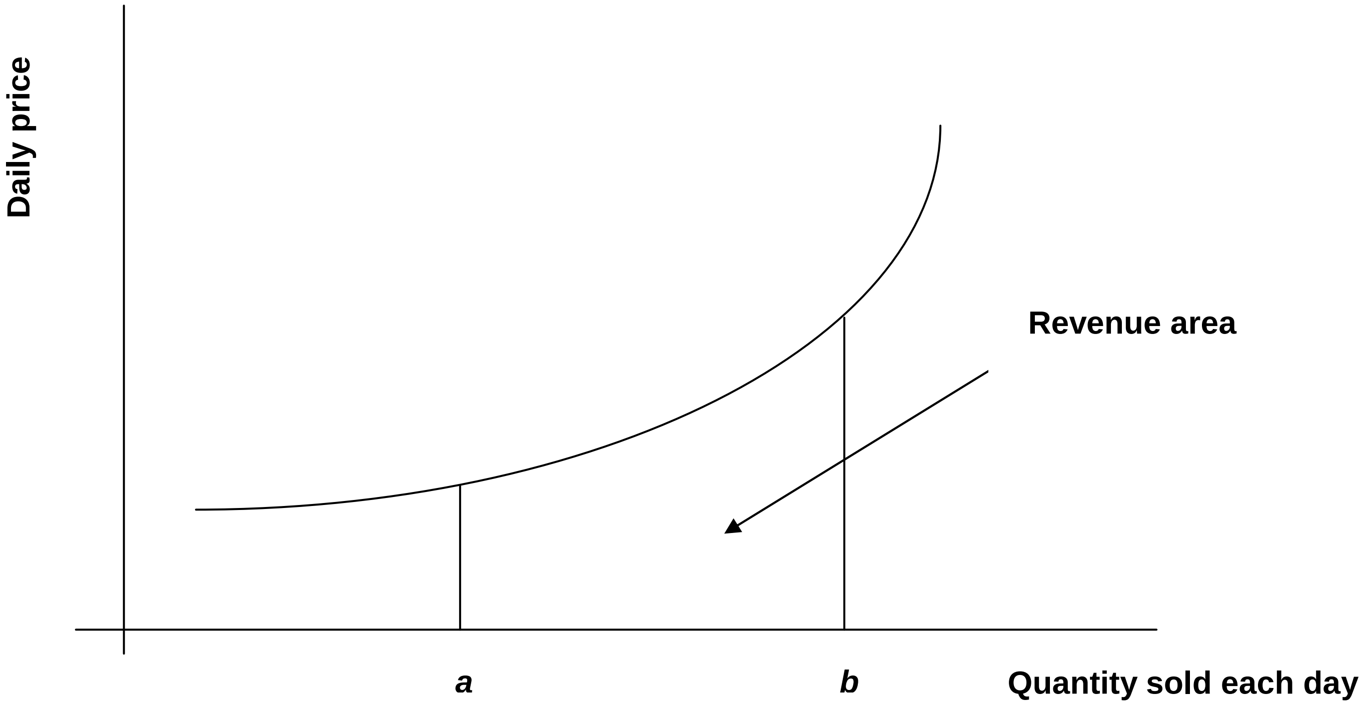
Students will be able to attempt this once they have a secure understanding of solving definite integrals. For example, lecturers might want to show students how integration can be used to calculate a Gini co-efficient or index:



There is also a clear direct application to consumer surplus (see the red area below). This could be a good way to consolidate earlier economics materials e.g. consumer surplus producer surplus, transfer earnings economic rent and so on.



Similarly, lecturers might want to create small case studies where definite integrals are required to solve a problem. For example, in the graph below the daily price of a commodity is graphed against the quantity sold each day. The total turnover over the given interval could be calculated using integration.



3. Delivering the concept of definite integrals to small or larger groups

Smaller groups could work through problems sets in pairs. Peer assessment could be useful here with students encouraged to give each other feedback on:

(a) how they solved the problem; and

(b) how they set out their work.

Many students grasp integration but make silly errors when completing the two part calculation. Simple peer assessment could help students to think about setting each step of their work minimising these small but significant slips.

Larger groups could benefit from more formal presentation on integration with accompanying worksheets. This work could be made more engaging by inviting students to mix and match integration problems with graphical representations.

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4. Discussion questions

Students could work in pairs or larger groups to think about the practical applications of integration. Students could be encouraged to review their notes from economics modules and to see practical applications e.g. a student who has studied microeconomics/ theory of the firm could see how definite integrals could be related to the calculations of revenues or costs. This could feed in to a wider discussion on the relevance of integration in economics. Pairs could summarise their thoughts and findings in a handout for their peers.

5. Activities

ACTIVITY ONE

Learning Objectives

LO1: Students to learn how to solve simple definite integrals

LO2: Students to consolidate understanding on integrating roots

LO3: Students to learn how to use integration to calculate the area under a graph

Task One

(i) Graph the following functions

(a) $y=3x^2$ for the range -5 to +5

(b) y=2x for the range -5 to +5

(c) $y=\frac{1}{x^2}$ for the range -5 to +5

(ii) Draw the following limits on your graphs

(a) 0 to +2

(b) +1 to +5

(c) +1 to +2

Task Two

Evaluate the following definite integrals:

(a) $\int_0^2 3x^2 \quad dx$

(b) $\int_1^5 2x - dx$

(c) $\int_1^2 \frac{1}{x^2} \quad dx$

ACTIVITY TWO

Learning Objectives

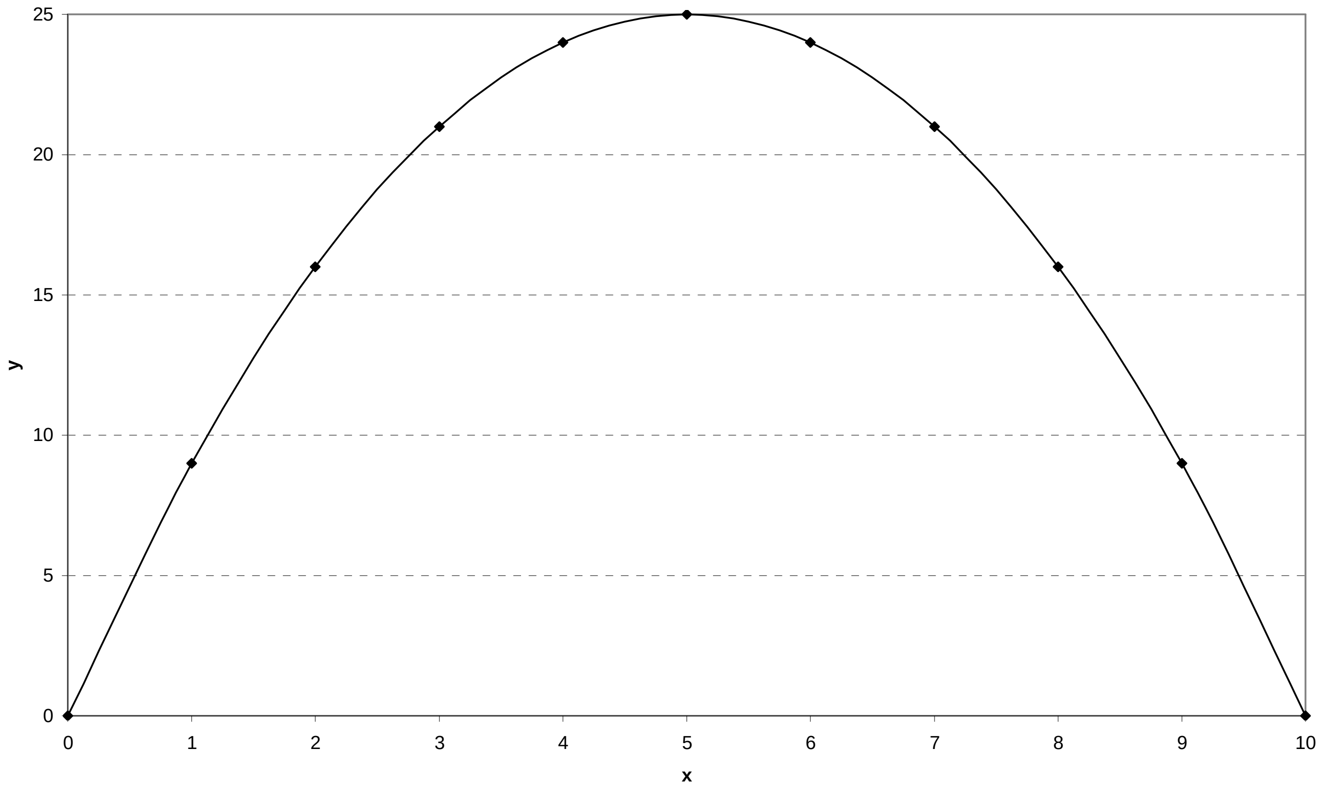
LO1: Students to learn how to calculate the area under a curve

LO2: Students to learn a practical application of integration to microeconomics

Task One

An economist undertakes some simple research on the relationship between profit (y) and quantity sold (x). He determines that:

$Y= 10x - x^2$ and Y and x have only positive values. The function is shown below



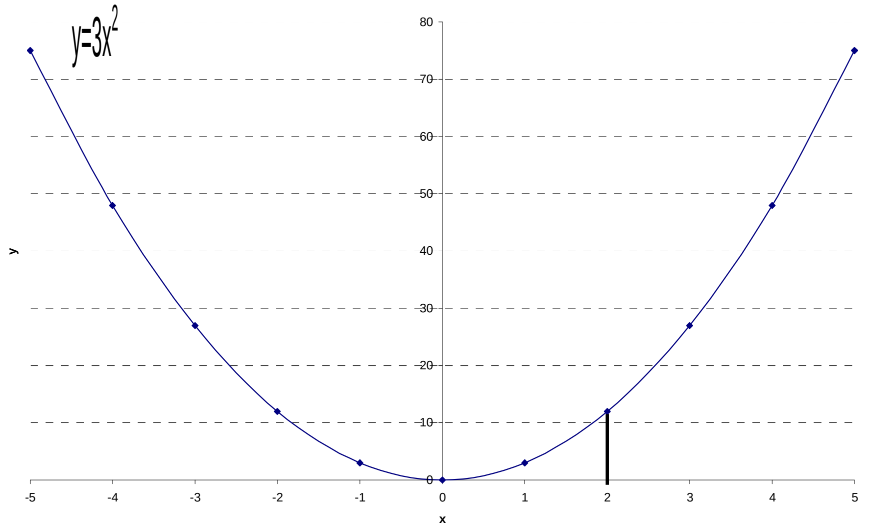
Calculate the area under this function between x=1 and x=10

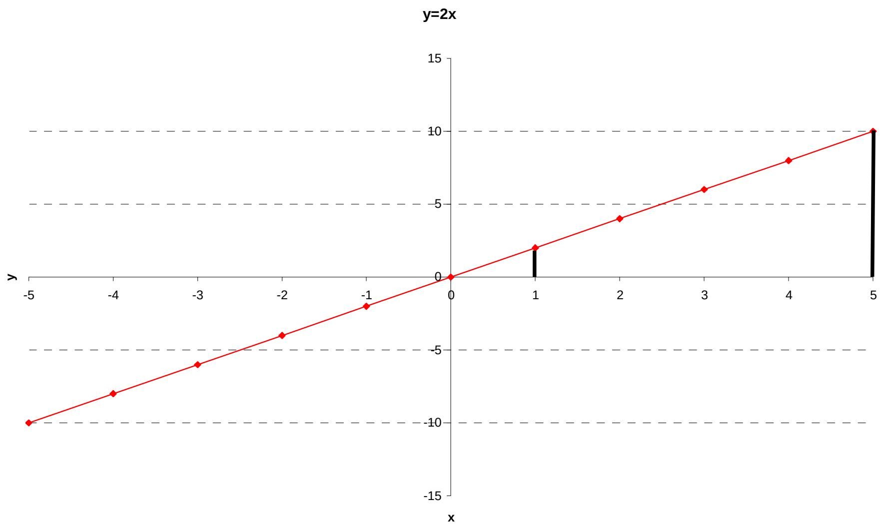
ANSWERS

ACTIVITY ONE

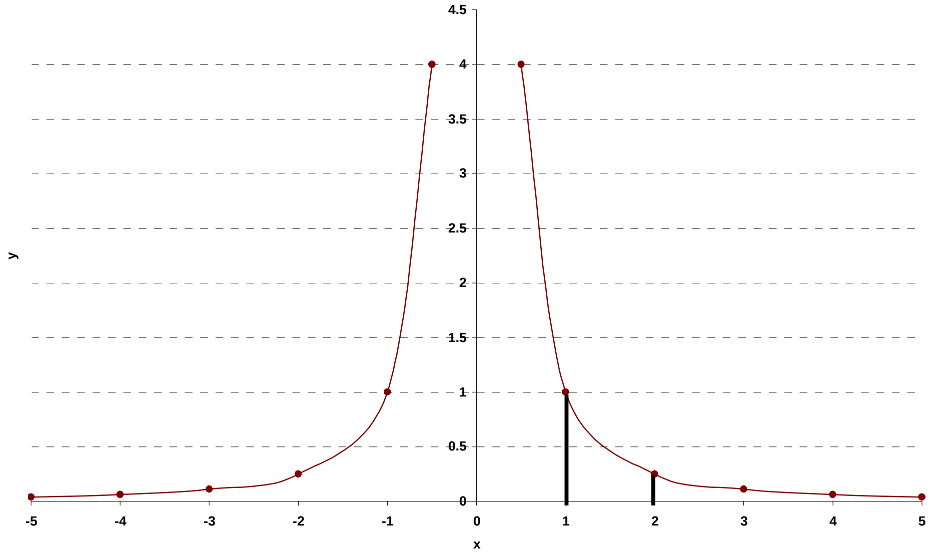
Task One

(i) and (ii)





y = $\frac{1}{x^2}$



Task Two

(a) 8

(b) 24

(c) 0.5

ACTIVITY TWO

Task One

Area = 162

6. Top Tips

Students can quickly lose interest with mathematics where there is little practical application or obvious context. Encourage students to think about integration can be used in 'real world' economics.

Students will need to practice the basic technique of solving definite integrals. Where possible, students could initially work in 'planned pairs' e.g. a higher ability students with one of lower ability. The higher ability can reinforce their learning by supporting their peer and the peer becomes an obvious beneficiary of the other students' higher ability.

Section 5: Integration and Exponential Functions

1. The concept of exponential functions and integration

This last section regarding integration and exponential functions is probably best left until last: students need to have a good grasp both of integration and the meaning and significance of *e* before they attempt this material. In some ways, this last section could be used as a plenary because it draws together a number of threads across a number of topics: calculus, logarithms and exponentials, indices and so on.

2. Presenting the concept of integrating exponential functions

Students will benefit from a basic restatement of what an exponential function is and the fact that it is based upon the constant *e*. Students might find it useful for some other 'universal constants' to be shared e.g. **pi** denoted Π, the **golden ratio** denoted by φ , i being equal to √-1, the speed of light in a vacuum (p equal to 299 792 458 m·s^{-1}). Students could be asked to prepare a 'factsheet' recording other universal constants.

3. Delivering the concept of integrating exponential functions to small or larger groups

All groups will need to have a brief reminder that:

$$\int e^x \, dx = e^x + c$$

and a simple example such as:

$$\int_1^3 e^x \, dx = [e^3]-[e^1]=20.085537 - 2.7182818$$

= $[e^3]-[e^1]=20.085537 - 2.7182818$

= 17.367

Students could be encouraged to explore the 'special property' of the *e* function such that any definite integral can be calculated using powers i.e. that $\int_1^3 e^x \, dx$ can be expressed as

$$e^{3-}e^{1} = e(e^{2-}1)$$

Larger groups could be shown a Powerpoint of exponential functions with a simple area under the curve highlighted to reinforce the idea of calculating areas. Smaller groups could be given straightforward exponential functions and asked to graph with pen and paper or using a spreadsheet programme such as Excel. This latter approach has the advantage of offering an opportunity for students to further explore the range of mathematical and statistical functions which Excel offers.

Both of these activities are designed to reinforce the idea of what an exponential functions, what it looks like and how it presents a 'special case' in integration.

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http://www.metalproject.co.uk/METAL/Resources/Question_bank/Calculus/index.html

Video clips

Teachers and lecturers might find it useful to review the video clips on differentiation before embarking on material concerning integration. These clips can be downloaded from: http://www.metalproject.co.uk/Resources/Films/Differential_equations/index.html. There are no clips which relate explicitly to integration although video clips can be used from other complementary areas to help lecturers deliver integration material. For example, some of the clips on supply and demand (see http://www.metalproject.co.uk/Resources/Films/Linear_equations/index.html) can easily be linked to the concept of consumer and producer surplus and so to the concept of integration and the area under a curve. The concept of an exponential function might be usefully linked to topics concerning finance and growth as well as more elementary material covered in Guide 1.

4. Discussion questions

This section could be developed by asking students to independently research where exponential functions might occur in business and economics e.g. rates of economic growth. Higher ability students could make inferences about particular contexts e.g. a pharmaceutical company might model bacterial growth using exponentials when researching a new antidote, nuclear power-plants would want to look at the cost of housing radioactive waste which might deteriorate exponentially, urban population growth might follow an exponential pattern which a wide range of economic effects upon employment, inflation, demand for public services etc.

The mathematical process of integrating exponential functions is unlikely to prompt discussion but researching and understanding its practical application offers a wealth of opportunity for students to explore 'real world' mathematics.

5. Activities

Learning Objectives

LO1: Students to practice integration of exponential functions

LO2: Students to consolidate earlier learning of expanding expressions

Task One

Evaluate:

(a) $\int_{2}^{3} e^{x} \, dx$

(b) $\int_{0}^{1} (3+e^{x})(2+e^{-x}) - dx$

(c) $\int_{0}^{3} e^{-x} \, dx$

ANSWERS

Task One

(a) 12.696

(b) 12.33

(c) 0.95

6. Top Tips and Conclusion

Students will need to have reviewed the meaning and significance of exponentials and the basics of integration before they attempt this section. Students will need to practice the basic operations as well as gaining some insights into how they can be applied to the solving of economic problems. Higher ability students could be givens cope to undertake independent research e.g. looking at (exponential) population growth in the developing world.